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End-of-Semester Assessment paper

Module Code: MH 4737
Module Title: Complex Analysis
Lecturer: Dr. B. Kreussler
External Examiner: Dr Sinéad Breen
Semester: Autumn 2012/2013
Duration of exam: Two hours
Percentage of total marks: 75%
Authorised Materials: Calculator, Mathematical tables

Instructions to candidates: Answer **three** of the following **five** questions.

Question 1

9 marks (a) Let $\gamma(t) = i + t(1-i)$, $0 \leq t \leq 2$, be a parametrisation of a line segment. Evaluate

$$\int_\gamma z^2 \, \mathrm{d}z.$$

7 marks (b) Use the **cross-ratio** to decide whether or not the four points

$$12+6i, \quad 4+10i, \quad -5+13i, \quad 18+2i$$

are on a circline.

9 marks (c) Determine the **residue** $\operatorname{res}\left(\frac{2z^3+3z^2}{(z-3)^2}, 3\right)$.

Question 2

7 marks (a) Calculate the **derivative** $f'(z)$ and write the first four non-zero terms of $f'(z)$, where

$$f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)2^k} z^{2k+1}.$$

10 marks (b) Let $U \subset \mathbb{C}$ be an open set, $f : U \to \mathbb{C}$ a holomorphic function, $a \in U$ and $\overline{D}(a,r) \subset U$ a closed disc. Let $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$, be a parametrisation of the boundary of this disc.

Prove that for all $z_0 \in D(a,r)$ *Cauchy's Integral Formula* holds:

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} \mathrm{d}z.$$

State, without proof, any theorems, lemmas etc. you are using in your proof.

8 marks (c) Let $\gamma(t) = 2i + e^{it}$, $0 \leq t \leq 2\pi$, be a parametrisation of the circle with centre $2i$ and radius 1. **Use** *Cauchy's Integral Formula* to evaluate

$$\int_\gamma \frac{z^2-4}{z(z^2+4)} \, \mathrm{d}z.$$

Question 3

9 marks (a) Determine all **poles** and their **orders** of the function

$$f(z) = \frac{z}{\left(z^2+1\right)^2\left(z^2+(4-2i)z-(1+4i)\right)}.$$

11 marks (b) Use the **generalised** Cauchy Integral Formula to evaluate

$$\int_\gamma \frac{1}{(z^2+1)^3}\,\mathrm{d}z,$$

where $\gamma : [0, 2\pi] \to \mathbb{C}$ is given by $\gamma(t) = i + e^{it}$.

5 marks (c) **Prove** that $e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$.

Question 4

9 marks (a) Determine the **Laurent series** of $f(z) = \dfrac{1}{(z-i)^2(z+2i)}$ valid on a punctured disc centred at $z_0 = i$.

Find the **radius** of the punctured disc on which your Laurent series is valid.

Write down the **principal part** of your Laurent series.

10 marks (b) **State**, without proof, the *Residue Theorem* **and use** it to evaluate the following integral in which $\gamma(t) = 1 + 2i + 3e^{it}$, $0 \leq t \leq 2\pi$, parametrises the circle with centre $1 + 2i$ and radius 3.

$$\int_\gamma \frac{z}{(z^2+9)(z-1)}\,\mathrm{d}z$$

6 marks (c) **Prove** *Liouville's Theorem*:
If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded, it must be constant.

State, without proof, any theorems, lemmas etc. you are using in your proof.

Question 5

12 marks (a) Find **all complex numbers** z for which $\sin(z) = \dfrac{13}{5}$.

HINT: $\sin\left(\dfrac{\pi}{2} + k\pi\right) = (-1)^k$ and $\cos\left(\dfrac{\pi}{2} + k\pi\right) = 0$ for all $k \in \mathbb{Z}$.

13 marks (b) Use the **tools** of complex analysis to evaluate the following integral.

$$\int_0^{2\pi} \frac{1}{61 + 11\cos(x)}\,\mathrm{d}x.$$